



# Expression for the Perturbation of the Weighted Moore-Penrose Inverse

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**Abstract**—We consider the perturbation formula for the weighted Moore-Penrose inverse of a rectangular matrix and give an explicit expression for the weighted Moore-Penrose inverse of a perturbed matrix under the weakest rank condition. This explicit expression extends the earlier work of several authors. © 2000 Elsevier Science Ltd. All rights reserved.

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## 1. INTRODUCTION

In this paper, we use the following notations. Let  $C^{m \times n}$  be the set of  $m \times n$  matrices with complex entries. For a matrix  $A \in C^{m \times n}$ , let  $A^* \in C^{n \times m}$  be the conjugate transpose of  $A$ ,  $\text{rank}(A)$  the rank of  $A$ ,  $R(A)$  the range of  $A$ ,  $N(A)$  the null space of  $A$ ,  $I_k$  the identity matrix of order  $k$ .

The weighted Moore-Penrose inverse of an arbitrary matrix (including singular and rectangular cases) has many applications in statistics, prediction theory, control systems and analysis, curve fitting, and numerical analysis [1–7].

For an arbitrary matrix  $A \in C^{m \times n}$  and Hermitian positive definite matrices  $M$  and  $N$  of order  $m$  and  $n$ , respectively, there is a unique matrix  $G \in C^{n \times m}$  satisfying the following equations:

$$AGA = A, \quad GAG = G, \quad (MAG)^* = MAG, \quad (NGA)^* = NGA. \quad (1.1)$$

$G$  is known as the weighted Moore-Penrose inverse of  $A$  and denoted by  $G = A_{MN}^+$ . In particular, when  $M = I_m$  and  $N = I_n$ , the matrix  $G$  satisfying (1.1) is called the Moore-Penrose inverse and denoted by  $G = A^+$ .

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The weighted Moore-Penrose inverse  $A_{MN}^+$  can be explicitly expressed by the weighted singular value decomposition due to Van Loan [3].

LEMMA 1.1. *Let  $A \in C^{m \times n}$  with  $\text{rank}(A) = r$ . Let  $M$  and  $N$  be Hermitian positive definite matrices of order  $m$  and  $n$ , respectively. Then there exist  $U \in C^{m \times m}$ ,  $V \in C^{n \times n}$  satisfying  $U^*MU = I_m$  and  $V^*N^{-1}V = I_n$  such that*

$$A = U \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} V^*, \quad (1.2)$$

and the weighted Moore-Penrose inverse  $A_{MN}^+$  can be represented as

$$A_{MN}^+ = N^{-1}V \begin{pmatrix} D^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^*M, \quad (1.3)$$

where  $D = \text{diag}(\mu_1, \mu_2, \dots, \mu_r)$ ,  $\mu_1 \geq \mu_2 \geq \dots \geq \mu_r > 0$  and  $\mu_i^2$  is the nonzero eigenvalue of  $A^\#A$  and  $A^\# = N^{-1}A^*M$ .  $\mu_i$  is called the weighted singular values.

Partition  $U = (U_1 \ U_2)$  and  $V = (V_1 \ V_2)$  compatible with (1.2), then the following simple relations that will be used later hold:

$$\begin{aligned} A_{MN}^+ &= N^{-1}V_1D^{-1}U_1^*M, & A_{MN}^+A &= N^{-1}V_1V_1^*, & AA_{MN}^+ &= U_1U_1^*M, \\ I_n - A_{MN}^+A &= N^{-1}V_2V_2^*, & I_m - AA_{MN}^+ &= U_2U_2^*M. \end{aligned} \quad (1.4)$$

Let  $B = A + E \in C^{m \times n}$ , where  $E$  is often viewed as a perturbation to  $A$ . What one concerns in practice is how the weighted Moore-Penrose inverse  $B_{MN}^+$  depends on the size of  $E$ . Ben-Israel [8] and Chen [1] gave a perturbation result under somewhat stronger assumptions  $R(E) \subseteq R(A)$  and  $R(E^*) \subseteq R(A^*)$  for the Moore-Penrose inverse and weighted Moore-Penrose inverse, respectively. Recently, Wei and Ding [4] extended their results to Hilbert spaces only assuming  $R(E) \subseteq R(A)$  or  $R(E^*) \subseteq R(A^*)$ . Chen and Xue [9] presented the expression of the Moore-Penrose inverse under the assumption that,  $\|A^+\| \|E\| < 1$  and  $\dim N(B) = \dim N(A)$ . In this note, we shall give an explicit expression of  $B_{MN}^+$  in terms of  $A_{MN}^+$  and  $E$  under the weakest rank condition that  $\text{rank}(A) = \text{rank}(B)$  and  $I_n + A_{MN}^+E$  is nonsingular. As byproduct, we obtain an explicit expression of  $B^+$  for the general case which extends the results by several authors [4,8,10–16].

## 2. MAIN RESULTS

Let  $B = A + E \in C^{m \times n}$ . We know that  $\text{rank}(B) = \text{rank}(A)$  is the necessary and sufficient condition for the continuity of Moore-Penrose inverse and weighted Moore-Penrose inverse. Otherwise, a small perturbation  $E$  would lead to a big difference between  $A_{MN}^+$  ( $A^+$ ) and  $B_{MN}^+$  ( $B^+$ ), see [17,18]. We first present an equivalent condition for  $\text{rank}(B) = \text{rank}(A)$ .

LEMMA 2.1. *Let  $A \in C^{m \times n}$  with  $\text{rank}(A) = r$ . Let  $B = A + E$  be such that  $I_n + A_{MN}^+E$  is nonsingular. Then  $\text{rank}(B) = \text{rank}(A)$  is equivalent to*

$$(I_m - AA_{MN}^+)E(I_n + A_{MN}^+E)^{-1}(I_n - A_{MN}^+A) = 0, \quad (2.1)$$

or

$$(I_m - AA_{MN}^+)(I_m + EA_{MN}^+)^{-1}E(I_n - A_{MN}^+A) = 0. \quad (2.2)$$

PROOF. Denote  $L = U^*MBN^{-1}V$ . From Lemma 1.1,  $L$  can be written as

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = \begin{pmatrix} D + P_1Q_1 & P_1Q_2 \\ P_2Q_1 & P_2Q_2 \end{pmatrix}, \quad (2.3)$$

where

$$P_i = U_i^* M E, \quad Q_i = N^{-1} V_i, \quad (i = 1, 2). \quad (2.4)$$

By (1.4)

$$I_n + A_{MN}^+ E = I_n + N^{-1} V_1 D^{-1} U_1^* M E = I_n + Q_1 D^{-1} P_1. \quad (2.5)$$

Thus, if  $I_n + A_{MN}^+ E$  is nonsingular, then so is  $D + P_1 Q_1$ . Therefore,  $\text{rank}(B) = \text{rank}(A)$  is equivalent to

$$\text{rank}(L) = \text{rank}(D + P_1 Q_1) = \text{rank}(L_{11}). \quad (2.6)$$

It follows from [19, Theorem 19] that

$$\text{rank}(L) = \text{rank}(D + P_1 Q_1) + \text{rank} \left[ P_2 Q_2 - P_2 Q_1 (D + P_1 Q_1)^{-1} P_1 Q_2 \right]. \quad (2.7)$$

Combining (2.6) and (2.7), we see that  $\text{rank}(B) = \text{rank}(A)$  is equivalent to

$$P_2 Q_2 - P_2 Q_1 (D + P_1 Q_1)^{-1} P_1 Q_2 = 0. \quad (2.8)$$

Multiplying (2.8) by  $U_2$  on the left and by  $V_2^*$  on the right, (2.8) is equivalent to

$$(I_m - A A_{MN}^+) E \left[ I_n - N^{-1} V_1 (D + U_1^* M E N^{-1} V_1)^{-1} U_1^* M E \right] (I_n - A_{MN}^+ A) = 0. \quad (2.9)$$

On the use of the Sherman-Morrison-Woodburg formula [11], we have

$$\begin{aligned} N^{-1} V_1 (D + U_1^* M E N^{-1} V_1)^{-1} U_1^* M &= N^{-1} V_1 \left[ D^{-1} - D^{-1} U_1^* M E (I_n + N^{-1} V_1 D^{-1} U_1^* M E)^{-1} \right. \\ &\quad \left. \times N^{-1} V_1 D^{-1} \right] U_1^* M \\ &= A_{MN}^+ - A_{MN}^+ E (I_n + A_{MN}^+ E)^{-1} A_{MN}^+ \\ &= (I_n + A_{MN}^+ E)^{-1} A_{MN}^+. \end{aligned} \quad (2.10)$$

Substituting (2.10) into (2.9), we get

$$(I_m - A A_{MN}^+) E \left[ I_n - (I_n + A_{MN}^+ E)^{-1} A_{MN}^+ E \right] (I_n - A_{MN}^+ A) = 0,$$

i.e.,

$$(I_m - A A_{MN}^+) E (I_n + A_{MN}^+ E)^{-1} (I_n - A_{MN}^+ A) = 0.$$

Notice that  $E(I_n + A_{MN}^+ E) = (I_m + E A_{MN}^+) E$ , the equivalence of (2.1) and (2.2) is obvious and we finish the proof.  $\blacksquare$

We are in position to give the main result of this section.

**THEOREM 2.2.** *Let  $B = A + E \in C^{m \times n}$  with  $\text{rank}(B) = \text{rank}(A) = r$ . Let  $M$  and  $N$  be Hermitian positive definite matrices of order  $m$  and  $n$ , respectively. Assume that  $I_n + A_{MN}^+ E$  is nonsingular. Then*

$$\begin{aligned} B_{MN}^+ &= (A_{MN}^+ A + N^{-1} X^* N) \left[ A_{MN}^+ A - X (N + X^* N X)^{-1} X^* N \right] (I_n + A_{MN}^+ E)^{-1} A_{MN}^+ \\ &\quad \times \left[ A A_{MN}^+ - M^{-1} Y^* (M^{-1} + Y M^{-1} Y^*)^{-1} Y \right] (A A_{MN}^+ + M^{-1} Y^* M), \end{aligned} \quad (2.11)$$

where  $X = (I_n + A_{MN}^+ E)^{-1} A_{MN}^+ E (I_n - A_{MN}^+ A)$  and  $Y = (I_m - A A_{MN}^+) E A_{MN}^+ (I_m + E A_{MN}^+)^{-1}$ .

**PROOF.** It can be easily verified that  $B_{MN}^+ = N^{-1} V L^+ U^* M$ , where  $L$  is given by (2.3). Due to [13, p. 34], we have

$$\begin{aligned} L^+ &= \begin{bmatrix} L_{11}^* \\ L_{12}^* \end{bmatrix} (L_{11}^* + L_{11}^{-1} L_{12} L_{12}^*)^{-1} L_{11}^{-1} (L_{11}^* + L_{21}^* L_{21} L_{11}^{-1})^{-1} [L_{11}^* \ L_{21}^*] \\ &= \begin{bmatrix} I_r \\ (L_{11}^{-1} L_{12})^* \end{bmatrix} \left[ I_r + L_{11}^{-1} L_{12} (L_{11}^{-1} L_{12})^* \right]^{-1} \\ &\quad \times L_{11}^{-1} \left[ I_r + (L_{21} L_{11}^{-1})^* L_{21} L_{11}^{-1} \right]^{-1} \left[ I_r \ (L_{21} L_{11}^{-1})^* \right]. \end{aligned}$$

Denote  $F = L_{11}^{-1}L_{12}$  and  $G = L_{21}L_{11}^{-1}$ . Then

$$L^+ = \begin{bmatrix} I_r \\ F^* \end{bmatrix} (I_r + FF^*)^{-1} L_{11}^{-1} (I_r + G^*G)^{-1} \begin{bmatrix} I_r & G^* \end{bmatrix}.$$

Hence, from the fact  $V_1^*N^{-1}V_1 = U_1^*MU_1 = I_r$  and (1.4), we have

$$\begin{aligned} B_{MN}^+ &= (N^{-1}V_1 + N^{-1}V_2F^*) (I_r + FF^*)^{-1} L_{11}^{-1} (I_r + G^*G)^{-1} (U_1^*M + G^*U_2^*M) \\ &= (N^{-1}V_1 + N^{-1}V_2F^*) V_1^*N^{-1}V_1 (I_r + FF^*)^{-1} V_1^*N^{-1}V_1 L_{11}^{-1} U_1^*MU_1 (I_r + G^*G)^{-1} \\ &\quad \times U_1^*MU_1 (U_1^*M + G^*U_2^*M) \\ &= (A_{MN}^+A + N^{-1}V_2F^*V_1^*) N^{-1}V_1 (I_r + FF^*)^{-1} V_1^*N^{-1}V_1 L_{11}^{-1} U_1^*M \\ &\quad \times U_1 (I_r + G^*G)^{-1} U_1^*M (AA_{MN}^+ + U_1G^*U_2^*M) \\ &= (A_{MN}^+A + B_2) B_4B_1B_5 (AA_{MN}^+ + B_3), \end{aligned} \quad (2.12)$$

where  $B_1 = N^{-1}V_1L_{11}^{-1}U_1^*M$ ,  $B_2 = N^{-1}V_2F^*V_1^*$ ,  $B_3 = U_1G^*U_2^*M$ ,  $B_4 = N^{-1}V_1(I_r + FF^*)^{-1}V_1^*$ , and  $B_5 = U_1(I_r + G^*G)^{-1}U_1^*M$ .

We now compute  $B_1$  to  $B_5$  individually. By (2.10) we get

$$B_1 = (I_n + A_{MN}^+E)^{-1} A_{MN}^+ = A_{MN}^+ (I_m + EA_{MN}^+)^{-1}. \quad (2.13)$$

Since

$$\begin{aligned} F &= L_{11}^{-1}L_{12} = V_1^*N^{-1}V_1L_{11}^{-1}U_1^*MU_1U_1^*MEN^{-1}V_2 \\ &= V_1^* (I_n + A_{MN}^+E)^{-1} A_{MN}^+AA_{MN}^+EN^{-1}V_2 \\ &= V_1^* (I_n + A_{MN}^+E)^{-1} A_{MN}^+EN^{-1}V_2. \end{aligned} \quad (2.14)$$

Similarly,

$$G = U_2^*MEA_{MN}^+ (I_m + EA_{MN}^+)^{-1} U_1. \quad (2.15)$$

Therefore, for  $B_2$  and  $B_3$ , we have

$$\begin{aligned} B_2 &= N^{-1}V_2V_2^*N^{-1} (A_{MN}^+E)^* \left[ I_n + (A_{MN}^+E)^* \right]^{-1} V_1V_1^* \\ &= N^{-1} (I_n - A_{MN}^+A)^* (A_{MN}^+E)^* \left[ I_n + (A_{MN}^+E)^* \right]^{-1} (A_{MN}^+A)^* N \\ &= N^{-1} (I_n - A_{MN}^+A)^* \left[ I_n + (A_{MN}^+E)^* \right]^{-1} (A_{MN}^+E)^* (A_{MN}^+A)^* N \\ &= N^{-1} (I_n - A_{MN}^+A)^* (A_{MN}^+E)^* \left[ I_n + (A_{MN}^+E)^* \right]^{-1} N \\ &= N^{-1}X^*N, \end{aligned} \quad (2.16)$$

where  $X = (I_n + A_{MN}^+E)^{-1}A_{MN}^+E(I_n - A_{MN}^+A)$  and

$$\begin{aligned} B_3 &= M^{-1} \left[ (I_m - AA_{MN}^+) EA_{MN}^+ (I_m + EA_{MN}^+)^{-1} \right]^* M \\ &= M^{-1}Y^*M, \end{aligned} \quad (2.17)$$

where  $Y = (I_m - AA_{MN}^+)EA_{MN}^+(I_m + EA_{MN}^+)^{-1}$ . Since

$$\begin{aligned} FF^* &= V_1^* (I_n + A_{MN}^+E)^{-1} A_{MN}^+EN^{-1}V_2V_2^*N^{-1} \left[ (I_n + A_{MN}^+E)^{-1} A_{MN}^+E \right]^* V_1 \\ &= V_1^* (I_n + A_{MN}^+E)^{-1} A_{MN}^+E (I_n - A_{MN}^+A) N^{-1} \left[ (I_n + A_{MN}^+E)^{-1} A_{MN}^+E \right]^* V_1 \\ &= V_1^* (I_n + A_{MN}^+E)^{-1} A_{MN}^+E (I_n - A_{MN}^+A) N^{-1} (I_n - A_{MN}^+A)^* \\ &\quad \times \left[ (I_n + A_{MN}^+E)^{-1} A_{MN}^+E \right]^* V_1 \\ &= V_1^*XN^{-1}X^*V_1. \end{aligned} \quad (2.18)$$

Likewise,

$$G^*G = U_1^*Y^*MYU_1. \quad (2.19)$$

By using of the Sherman-Morrison-Woodburg formula [11] again, we have

$$\begin{aligned} (I_r + FF^*)^{-1} &= (I_r + V_1^*XN^{-1}X^*V_1)^{-1} \\ &= I_r - V_1^*XN^{-1}(I_n + X^*V_1V_1^*XN^{-1})^{-1}X^*V_1 \\ &= I_r - V_1^*X(N + X^*NX)^{-1}X^*V_1. \end{aligned} \quad (2.20)$$

As for  $B_4$ , we obtain

$$B_4 = A_{MN}^+A - X(N + X^*NX)^{-1}X^*N. \quad (2.21)$$

Finally,

$$B_5 = AA_{MN}^+ - M^{-1}Y^*(M^{-1} + YM^{-1}Y^*)^{-1}Y. \quad (2.22)$$

Substituting equations (2.13), (2.16), (2.17), (2.21), and (2.22) in (2.12) leads to (2.11). The proof is over. ■

From Theorem 2.2, we can immediately obtain the following corollaries.

**COROLLARY 2.3.** *Let  $B = A + E \in C^{m \times n}$  with  $\text{rank}(B) = \text{rank}(A)$ . If  $I_n + A^+E$  is nonsingular, then*

$$\begin{aligned} B^+ &= (A^+A + X^*) \left[ A^+A - X(I_n + X^*X)^{-1}X^* \right] (I_n + A^+E)^{-1}A^+ \\ &\quad \times \left[ AA^+ - Y^*(I_m + YY^*)^{-1}Y \right] (AA^+ + Y^*), \end{aligned} \quad (2.23)$$

where  $X = (I_n + A^+E)^{-1}A^+E(I_n - A^+A)$  and  $Y = (I_m - AA^+)EA^+(I_m + EA^+)^{-1}$ .

This explicit expression for the perturbation of Moore-Penrose inverse is different from Theorem 1 in [9].

**COROLLARY 2.4.** *Let  $B = A + E \in C^{m \times n}$  and  $I_n + A_{MN}^+E$  be invertible.*

(i) *If  $R(E^*) \subseteq R(A^*)$ , then*

$$\begin{aligned} B_{MN}^+ &= (I_n + A_{MN}^+E)^{-1}A_{MN}^+ \\ &\quad \left[ AA_{MN}^+ - M^{-1}Y^*(M^{-1} + YM^{-1}Y^*)^{-1}Y \right] (AA_{MN}^+ + M^{-1}Y^*M). \end{aligned} \quad (2.24)$$

(ii) *If  $R(E) \subseteq R(A)$ , then*

$$B_{MN}^+ = (A_{MN}^+A + N^{-1}X^*N) \left[ A_{MN}^+A - X(N + X^*NX)^{-1}X^*N \right] (I_n + A_{MN}^+E)^{-1}A_{MN}^+. \quad (2.25)$$

(iii) *If  $R(E) \subseteq R(A)$  and  $R(E^*) \subseteq R(A^*)$ , then*

$$B_{MN}^+ = (I_n + A_{MN}^+E)^{-1}A_{MN}^+ = A_{MN}^+ (I_m + EA_{MN}^+)^{-1}. \quad (2.26)$$

**PROOF.** By Lemma 2.1  $\text{rank}(B) = \text{rank}(A)$  holds. If  $R(E^*) \subseteq R(A^*)$ , we have  $E(I_n - A_{MN}^+A) = 0$ , which implies  $X = 0$ . If  $R(E) \subseteq R(A)$ , then  $(I_m - AA_{MN}^+)E = 0$ , and thus,  $Y = 0$ . Hence, (i)–(iii) follow directly from Theorem 2.2. ■

**COROLLARY 2.5.** *Let  $B = A + E \in C^{m \times n}$  and  $I_n + A^+E$  be invertible.*

(i) *If  $R(E^*) \subseteq R(A^*)$ , then*

$$B^+ = (I_n + A^+E)^{-1}A^+ \left[ AA^+ - Y^*(I_m + YY^*)^{-1}Y \right] (AA^+ + Y^*). \quad (2.27)$$

(ii) *If  $R(E) \subseteq R(A)$ , then*

$$B^+ = (A^+A + X^*) \left[ A^+A - X(I_n + X^*X)^{-1}X^* \right] (I_n + A^+E)^{-1}A^+. \quad (2.28)$$

(iii) *If  $R(E) \subseteq R(A)$  and  $R(E^*) \subseteq R(A^*)$ , then*

$$B^+ = (I_n + A^+E)^{-1}A^+ = A^+ (I_m + EA^+)^{-1}. \quad (2.29)$$

We mention that (iii) of Corollary 2.4 and Corollary 2.5 are the generalization of the result of [1,4,8], respectively.

### 3. REMARKS

A highly accurate computation of the singular values of a matrix is a topic of current interest in the literature. Dilenia, Peluso and Piazza [12] developed bounds on relative perturbation of singular values as follows.

Let  $A, B \in C^{m \times n}$ . For any singular values  $\sigma_i(A) > 0$ , if  $R(B) \subseteq R(A)$ , then  $|(\sigma_i(A) - \sigma_i(B))/\sigma_i(A)| \leq \|A^+(A - B)\|$ ; if  $R(B^*) \subseteq R(A^*)$ , then  $|(\sigma_i(A) - \sigma_i(B))/\sigma_i(A)| \leq \|(A - B)A^+\|$ .

We can easily extend these results to the weighted singular values under the same assumption. It is of interest to establish the general bounds on the relative perturbation of (weighted) singular values under the hypothesis,  $\text{rank}(A) = \text{rank}(B)$ .

### REFERENCES

1. G.L. Chen, Minimum property of weighted condition number in matrix perturbation problems, *J. East China Normal University* **1**, 1-7, (1992).
2. W.Y. Sun and Y.M. Wei, Inverse order rule for weighted generalized inverse, *SIAM J. Matrix Anal. Appl.* **19**, 772-775, (1998).
3. C.F. Van Loan, Generalizing the singular value decomposition, *SIAM J. Numer. Anal.* **13**, 76-83, (1976).
4. Y.M. Wei and J. Ding, Representation for Moore-Penrose inverse in Hilbert space, (preprint).
5. Y.M. Wei, Solving singular linear system and generalized inverse, Ph.D Thesis, Institute of Mathematics, Fudan University, Shanghai, China, (1997).
6. Y.M. Wei, Perturbation bound of singular linear system, *Applied Math. & Comput.* **105**, 211-220, (1999).
7. Y.M. Wei, Recurrent neural networks for computing weighted Moore-Penrose inverse, *Applied Math. & Comput.* (to appear).
8. A. Ben-Israel, On error bounds for generalized inverses, *SIAM J. Numer. Anal.* **3**, 585-592, (1966).
9. G.L. Chen and Y.F. Xue, The expression of the generalized inverse of the perturbed operator under type I perturbation in Hilbert spaces, *Linear Algebra Appl.* **285**, 1-6, (1998).
10. R.E. Cline, Representations of the generalized inverse of sums of matrices, *SIAM J. Numer. Anal.* **2**, 99-114, (1965).
11. R.E. Cline and R.E. Funderlic, The rank of a difference of matrices and the associated generalized inverse, *Linear Algebra Appl.* **24**, 185-215, (1979).
12. G. Dilenia, R.I. Peluso and G. Piazza, Results on the relative perturbation of the singular values of a matrix, *BIT* **33**, 647-653, (1993).
13. R.E. Hartwig, Singular value decomposition and the Moore-Penrose inverse of bordered matrices, *SIAM J. Appl. Math.* **31**, 31-41, (1996).
14. C.D. Meyer, Generalized inversion of modified matrices, *SIAM J. Appl. Math.* **24**, 315-323, (1973).
15. N. Minamide, An extension of the matrix inversion lemma, *SIAM J. Alg. Disc. Math.* **6**, 371-377, (1985).
16. K. Radoslaw and K. Krezstef, Generalized inverses of a sum of matrices, *Sankhya Ser. A* **56**, 458-464, (1994).
17. G.W. Stewart, On the perturbation of pseudo-inverse, projections, and linear least squares problems, *SIAM Rev.* **19**, 634-662, (1977).
18. P.A. Wedin, Perturbation theory for pseudoinverses, *BIT* **13**, 217-232, (1973).
19. G. Marsaglia and G.P.H. Styan, Equalities and inequalities for ranks of matrices, *Linear and Multilinear Algebra* **2**, 269-292, (1974).